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Renormalization group evolution of the CKM matrix

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Abstract. We compute the renormalization of the complete CKM matrix in the $\overline{MS}$ scheme and perform a renormalization group analysis of the CKM parameters. The calculation is simplified by studying only the Higgs sector, which for the β-function of the CKM matrix is at one loop the same as in the full Standard Model. The renormalization group flow including QCD corrections can be computed analytically using the hierarchy of the CKM parameters and the large mass differences between the quarks. While the evolution of the Cabibbo angle is tiny V_{ub} and V_{cb} increase sizably. We compare our results with the ones in the full Standard Model.

1 Introduction

In the Standard Model (SM) and in all possible extensions the origin of flavor mixing lies in the Higgs sector and thus belongs to its least understood part. While in the quark sector this phenomenon is parametrized by the Cabibbo-Kobayashi-Maskawa (CKM) matrix [1], it is still not clear whether a similar effect exists for the leptons. Here mixing can only happen once the neutrinos have masses, for which recently some evidence has been given [2].

As a first step in understanding the origin of the CKM matrix it is useful to compute its renormalization group evolution, since one may hope that some unknown physics fixes a CKM matrix or, equivalently, mass matrices for quarks and leptons at a high scale Λ. Thus its structure can give some hint on the overlying theory which produces this CKM matrix as an effective coupling. The CKM matrix elements are measured at hadronic scales of a few GeV, or maybe at the electroweak scale if they are extracted from W decays one day.

Studies of the renormalization of the CKM matrix can already be found in the literature. The one loop contributions to the CKM matrix have been computed in [3] in the on-shell scheme. It has been found that the corrections are small and hence they have been ignored in all analysis. However, in [3] the renormalization group flow has not been investigated. In a couple of other papers [4–8] the renormalization group flow of the parameters of the Higgs sector has been studied. Since the system of renormalization group equations is quite complicated these studies have been performed numerically. If one considers non-SM scenarios mixing can also occure in the leptonic sector which has been studied in [9].

In the present paper we present a renormalization group study of the CKM matrix, where we simplify matters in such a way that we may even construct an analytic solution of the renormalization group equations which is an excellent approximation below the GUT scale of 10^{15} GeV. Thus we restrict ourselves to a one-loop analysis in the limit of vanishing elektroweak gauge couplings. Consequently for the renormalization of the CKM matrix only the Higgs sector remains but, as we shall see, the full SM result is reproduced at one loop.

In the next section we shall "ungauge" the elektroweak part of the SM by taking the limit of vanishing gauge coupling in an appropriate way. Since the renormalization group evolution of the quark masses is mainly driven by strong interactions, the QCD part remains as in the full SM. Section 3 discusses the renormalization of this pure Higgs sector. In particular it is shown that due to a Ward identity the renormalization of the CKM matrix only involves the wave function renormalization matrices of the left handed quarks. This is true to all orders in the loop expansion. In Sect. 4 we formulate the renormalization group equations for the CKM parameters and the masses and solve them analytically in a certain approximation. In Sect. 5 we check the quality of our analytic solution by comparing with the full SM, i.e. with non-vanishing electroweak couplings. Finally we discuss our results and conclude.

2 Higgs sector and flavor mixing

Flavor mixing is entirely generated by the Higgs sector and the physics of this effect should be understandable without the complications of the gauge theory. Thus we choose to "ungauge" $SU(2) \otimes U(1)_Y$ in the following way. We take the limit $g_1 \to 0$ and $g_2 \to 0$ (g_1 and g_2 being the $SU(2)$ and $U(1)_Y$ couplings respectively) keeping the vacuum expectation value of the Higgs field fixed. Furthermore, the ratio g_1/g_2 defining the weak mixing angle

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does not enter our consideration. In this limit the longitudinal modes of the weak bosons appear as massless scalar fields, namely as the Goldstone bosons of the spontaneously broken $SU(2)_L$, while the transverse degrees of freedom decouple.

We shall group all known quarks and leptons into left and right handed doublets according to

$$L_u = \begin{pmatrix} u \\ d \end{pmatrix}_L \qquad L_c = \begin{pmatrix} c \\ s \end{pmatrix}_L \qquad L_t = \begin{pmatrix} t \\ b \end{pmatrix}_L \tag{1}$$

$$R_u = \begin{pmatrix} u \\ d \end{pmatrix}_R \qquad R_c = \begin{pmatrix} c \\ s \end{pmatrix}_R \qquad R_t = \begin{pmatrix} t \\ b \end{pmatrix}_R \tag{2}$$

$$L_e = \begin{pmatrix} \nu_e \\ e \end{pmatrix}_L \qquad L_\mu = \begin{pmatrix} \nu_\mu \\ \mu \end{pmatrix}_L \qquad L_\tau = \begin{pmatrix} \nu_\tau \\ \tau \end{pmatrix}_L \tag{3}$$

$$R_e = \begin{pmatrix} \nu_e \\ e \end{pmatrix}_R \qquad R_\mu = \begin{pmatrix} \nu_\mu \\ \mu \end{pmatrix}_R \qquad R_\tau = \begin{pmatrix} \nu_\tau \\ \tau \end{pmatrix}_R . \tag{4}$$

Note that we have also introduced right handed neutrino fields in order to complete the right handed leptonic doublets. We shall write the Higgs sector of the SM first as a linear sigma model which has a full $SU(2)_L \otimes SU(2)_R$ symmetry. Upon spontaneous breaking this symmetry is reduced to $SU(2)_{L+R}$ corresponding to the custodial symmetry of the SM. As we shall see, flavor mixing is related to the explicit breaking of this symmetry and hence we introduce some explicit breaking later.

Under this symmetry the left handed leptons and quarks transform as a $(2,0)$ while the right handed components are assigned to the $(0,2)$ representation. Furthermore, to make contact with the weak hypercharge of the SM we postulate another $U(1)$ symmetry under which we assign the following charges

$$\begin{aligned}
L_q &\xrightarrow{U(1)} e^{i\,(1/3)\,\omega} L_q \qquad (q = u,c,t) \\
L_l &\xrightarrow{U(1)} e^{i\,(-1)\,\omega} L_l \qquad (l = e,\mu,\tau) \\
R_q &\xrightarrow{U(1)} e^{i\,(1/3)\,\omega} R_q \qquad (q = u,c,t) \\
R_l &\xrightarrow{U(1)} e^{i\,(-1)\,\omega} R_l \qquad (l = e,\mu,\tau) .
\end{aligned} \tag{5}$$

The four Higgs fields are gathered in a 2×2 matrix according to

$$\mathbb{H} = \begin{pmatrix} \varphi_0 - i\chi & \sqrt{2}\,\phi^+ \\ -\sqrt{2}\,\phi^- & \varphi_0 + i\chi \end{pmatrix} \tag{6}$$

transforming in an obvious way under $(2,\bar{2})$ while it is invariant under this additional $U(1)$. The Higgs fields are governed by the standard lagrangian of the linear sigma model

$$\begin{aligned}
\mathcal{L} &= \frac{1}{2}\,\mathrm{Tr}\,\left[(\partial_\mu \mathbb{H}^\dagger)\,(\partial^\mu \mathbb{H})\right] - \frac{\lambda}{64}\left[\mathrm{Tr}\,(\mathbb{H}^\dagger\,\mathbb{H})\right]^2 \\
&\quad + \frac{\mu^2}{4}\mathrm{Tr}\,(\mathbb{H}^\dagger\,\mathbb{H})
\end{aligned} \tag{7}$$

which exhibits spontaneous symmetry breaking, if $\mu^2 > 0$. We choose the vacuum expectation value such that at tree level

$$\varphi_0 = v + H\,, \qquad v = \sqrt{\frac{4\mu^2}{\lambda}}\,. \tag{8}$$

This choice yields a breaking term proportional to the unit matrix which breaks the full $SU(2)_L \otimes SU(2)_R$ symmetry down to the diagonal $SU(2)_{L+R}$ which is usually called custodial $SU(2)$. The field H is the physical Higgs field while the other fields χ, $\phi^\pm$ are the Goldstone bosons and correspond to the longitudinal degrees of freedom of the Z_0 and $W^\pm$.

The only possible renormalizable coupling terms of the Higgs fields to the matter fields invariant under $SU(2)_L \otimes SU(2)_R$ and the additional $U(1)$ are

$$\begin{aligned}
\mathcal{L}^{(0)}_{ffH} &= -\sum_{A,B=u,c,t} \bar{L}_A \mathbb{H}\, R_B G_{AB} \\
&\quad - \sum_{a,b=e,\mu,\tau} \bar{L}_a \mathbb{H}\, R_b g_{ab} + \text{h.c.}\,.
\end{aligned} \tag{9}$$

Obviously the matrices of Yukawa couplings G_{AB} and g_{ab} can be diagonalized by the usual biunitary transformation without any effects on the other terms in the lagrangian and hence no flavor mixing can appear as long as the custodial $SU(2)_{L+R}$ remains unbroken.

Different masses for up- and down-type quarks as well as the coupling to hypercharge break the custodial $SU(2)$ in the full SM. In the "ungauged" model we introduce the breaking of $SU(2)_{L+R}$ by an additional coupling of the form:

$$\begin{aligned}
\mathcal{L}^{(1)}_{ffH} &= -\sum_{A,B=u,c,t} \bar{L}_A \mathbb{H}\, \sigma_3 R_B \tilde{G}_{AB} \\
&\quad - \sum_{a,b=e,\mu,\tau} \bar{L}_a \mathbb{H}\, \sigma_3 R_b \tilde{g}_{ab} + \text{h.c.}\ ,
\end{aligned} \tag{10}$$

which also breaks $SU(2)_R$ down to a $U(1)_R$. In this way the relation between the breaking of custodial $SU(2)$ and mixing becomes transparent.

The symmetry needed for the elektroweak part of the SM is still present as a combination of the $U(1)$ introduced above and this $U(1)_R$. Hence we introduce a hypercharge $U(1)_Y$, under which the fermion dubletts transform according to

$$\begin{aligned}
L_q &\xrightarrow{U(1)_Y} e^{i\,(1/3)\,\omega} L_q \qquad (q = u,c,t) \\
L_l &\xrightarrow{U(1)_Y} e^{i\,(-1)\,\omega} L_l \qquad (l = e,\mu,\tau) \\
R_q &\xrightarrow{U(1)_Y} e^{i\,(1/3+\sigma_3)\,\omega} R_q \qquad (q = u,c,t) \\
R_l &\xrightarrow{U(1)_Y} e^{i\,(-1+\sigma_3)\,\omega} R_l \qquad (l = e,\mu,\tau) .
\end{aligned} \tag{11}$$

Due to the explicit breaking of custodial $SU(2)$ the up and down type quarks aquire different mass matrices defined as

$$\begin{aligned}
G_{u,AC} &\equiv v\left(G_{AC} + \tilde{G}_{AC}\right) \\
G_{d,BD} &\equiv v\left(G_{BD} - \tilde{G}_{BD}\right).
\end{aligned} \tag{12}$$

In the following we shall discuss only quarks for which we introduce the compact notation

$$\mathbf{u}_{L,R} = \begin{pmatrix} u \\ c \\ t \end{pmatrix}_{L,R} \qquad \mathbf{d}_{L,R} = \begin{pmatrix} d \\ s \\ b \end{pmatrix}_{L,R} . \tag{13}$$

The mass terms for the quarks take the form

$$-\bar{\mathbf{u}}_L \mathbf{G}_u \mathbf{u}_R - \bar{\mathbf{u}}_R \mathbf{G}_u^\dagger \mathbf{u}_L - \bar{\mathbf{d}}_L \mathbf{G}_d \mathbf{d}_R - \bar{\mathbf{d}}_R \mathbf{G}_d^\dagger \mathbf{d}_L \tag{14}$$

which upon diagonalization yields the mass spectrum of the quarks

$$\begin{aligned} \mathbf{m}_u &= \mathbf{S}_u^{L\dagger}\, \mathbf{G}_u\, \mathbf{S}_u^R = \mathbf{diag}\ (m_u, m_c, m_t) \\ \mathbf{m}_d &= \mathbf{S}_d^{L\dagger}\, \mathbf{G}_d\, \mathbf{S}_d^R = \mathbf{diag}\ (m_d, m_s, m_b) \end{aligned} \tag{15}$$

where $m_i > 0$. As in the full SM the CKM matrix is given by

$$\mathbf{V} \equiv \mathbf{S}_u^{L\,\dagger}\, \mathbf{S}_d^L. \tag{16}$$

In this basis of mass eigenstates we find in the broken phase

$$\begin{aligned} \mathcal{L} = {} & \mathcal{L}_{kin}^{Higgs} + \mathcal{L}_{kin}^{quarks} + \mathcal{L}_{int}^{Higgs} \\ & + \mathcal{L}_{int}^{neutral} + \mathcal{L}_{int}^{charged} \end{aligned} \tag{17}$$

$$\begin{aligned} \mathcal{L}_{kin}^{Higgs} &= \frac{1}{2} H \left(\Box - M_H^2\right) H + \frac{1}{2}\chi \left(\Box - M_\chi^2\right)\chi \\ &\quad + \phi^+ \left(\Box - M_\chi^2\right)\phi^- \end{aligned} \tag{18}$$

$$\begin{aligned} \mathcal{L}_{kin}^{Quarks} &= \bar{\mathbf{u}}_L i\partial\!\!\!/ \mathbf{u}_L + \bar{\mathbf{u}}_R i\partial\!\!\!/ \mathbf{u}_R - \bar{\mathbf{u}}_L \mathbf{m}_u \mathbf{u}_R - \bar{\mathbf{u}}_R \mathbf{m}_u \mathbf{u}_L \\ &\quad + \bar{\mathbf{d}}_L i\partial\!\!\!/ \mathbf{d}_L + \bar{\mathbf{d}}_R i\partial\!\!\!/ \mathbf{d}_R \\ &\quad - \bar{\mathbf{d}}_L \mathbf{m}_d \mathbf{d}_R - \bar{\mathbf{d}}_R \mathbf{m}_d \mathbf{d}_L \end{aligned} \tag{19}$$

$$\begin{aligned} \mathcal{L}_{int}^{Higgs} &= -v M_\chi^2 H - \frac{M_H^2 - M_\chi^2}{2v}\left[H^3 + H\chi^2 + 2H\phi^+\phi^-\right] \\ &\quad - \frac{M_H^2 - M_\chi^2}{8v^2}\Big[H^4 + \chi^4 + 4\left(\phi^+\phi^-\right)^2 + 2H^2\chi^2 \\ &\quad + 4H^2\phi^+\phi^- + 4\chi^2\phi^+\phi^-\Big] \end{aligned} \tag{20}$$

$$\begin{aligned} \mathcal{L}_{int}^{neutral} &= -\frac{1}{v}\bar{\mathbf{u}}_L \mathbf{m}_u \mathbf{u}_R (H - i\chi) - \frac{1}{v}\bar{\mathbf{u}}_R \mathbf{m}_u \mathbf{u}_L (H + i\chi) \\ &\quad - \frac{1}{v}\bar{\mathbf{d}}_L \mathbf{m}_d\, \mathbf{d}_R (H - i\chi) \\ &\quad - \frac{1}{v}\bar{\mathbf{d}}_R \mathbf{m}_d\, \mathbf{d}_L (H + i\chi) \end{aligned} \tag{21}$$

$$\begin{aligned} \mathcal{L}_{int}^{charged} &= -\frac{\sqrt{2}}{v}\bar{\mathbf{u}}_L \mathbf{V} \mathbf{m}_d\, \mathbf{d}_R \phi^+ + \frac{\sqrt{2}}{v}\bar{\mathbf{u}}_R \mathbf{m}_u \mathbf{V} \mathbf{d}_L \phi^+ \\ &\quad + \frac{\sqrt{2}}{v}\bar{\mathbf{d}}_L \mathbf{V}^\dagger \mathbf{m}_u \mathbf{u}_R \phi^- - \frac{\sqrt{2}}{v}\bar{\mathbf{d}}_R \mathbf{m}_d \mathbf{V}^\dagger \mathbf{u}_L \phi^- . \end{aligned} \tag{22}$$

For the purpose of renormalization we have also introduced a mass for the Goldstone bosons such that the masses of the Higgs particles are

$$\begin{aligned} M_H^2 &\equiv \frac{3}{4}\lambda v^2 - \mu^2 \\ M_\chi^2 &\equiv \frac{1}{4}\lambda v^2 - \mu^2. \end{aligned} \tag{23}$$

At tree level $M_\chi^2 = 0$ but for renormalization it is advantageous to keep v as an independent parameter. Equation (21) represents the neutral currents and the interactions with the physical Higgs. These contributions - as in the full SM - do not induce quark mixing. The charge current interactions (22) involve the CKM matrix $\mathbf{V}$ and are the source of flavor mixing.

3 Renormalization

The Higgs sector is introduced in such a way that the full as well as the "ungauged" SM is renormalizable. We are aiming here at the one loop renormalization of the CKM matrix which can be obtained from the quark self energies only. This is due to a Ward identity which actually holds to all orders and even in the full SM. It is a consequence of $SU(2)_L$ and is derived in the unbroken phase. In this phase global $SU(2)_L$ is a manifest symmetry which translates into the Ward identities

$$\begin{aligned} \frac{\delta\Gamma}{\delta\phi_0^-}\,\varphi_{0,0} + \sqrt{2}\left[\frac{\delta\Gamma}{\delta\mathbf{d}_{L,0}}\mathbf{u}_{L,0} + \bar{\mathbf{d}}_{L,0}\frac{\delta\Gamma}{\delta\bar{\mathbf{u}}_{L,0}}\right] &= 0 \\ \frac{\delta\Gamma}{\delta\phi_0^+}\,\varphi_{0,0} - \sqrt{2}\left[\frac{\delta\Gamma}{\delta\mathbf{u}_{L,0}}\mathbf{d}_{L,0} + \bar{\mathbf{u}}_{L,0}\frac{\delta\Gamma}{\delta\bar{\mathbf{d}}_{L,0}}\right] &= 0 \end{aligned} \tag{24}$$

for the generating functional Γ of one particle irreducible Greensfunctions. The functions $\phi_0^\pm$ and $\mathbf{u}/\mathbf{d}_{L,0}$ have to be regarded as sources for the corresponding fields. (24) holds for the bare[1] Higgs field and bare elektroweak eigenstates and we have suppressed terms which will not appear as external states. In the broken phase the same Ward identities hold with $\varphi_{0,0}$ replaced by $v_0 + H_0$

$$\begin{aligned} \frac{\delta\Gamma}{\delta\phi_0^-}\,v_0 + \sqrt{2}\left[\frac{\delta\Gamma}{\delta\mathbf{d}_{L,0}}\mathbf{u}_{L,0} + \bar{\mathbf{d}}_{L,0}\frac{\delta\Gamma}{\delta\bar{\mathbf{u}}_{L,0}}\right] &= 0 \\ \frac{\delta\Gamma}{\delta\phi_0^+}\,v_0 - \sqrt{2}\left[\frac{\delta\Gamma}{\delta\mathbf{u}_{L,0}}\mathbf{d}_{L,0} + \bar{\mathbf{u}}_{L,0}\frac{\delta\Gamma}{\delta\bar{\mathbf{d}}_{L,0}}\right] &= 0\,. \end{aligned} \tag{25}$$

We have ommited the term with the physical Higgs field since it does not contribute to the renormalization of the charged current. Transforming to bare mass eigenstates according to

$$\begin{aligned} \mathbf{u}_{L/R,0} &\to \mathbf{S}_{u,0}^{L/R}\,\mathbf{u}_{L/R,0} \\ \mathbf{d}_{L/R,0} &\to \mathbf{S}_{d,0}^{L/R}\,\mathbf{d}_{L/R,0} \end{aligned} \tag{26}$$

we get

$$\begin{aligned} \frac{\delta\Gamma}{\delta\phi_0^-}\,v_0 + \sqrt{2}\left[\frac{\delta\Gamma}{\delta\mathbf{d}_{L,0}}\mathbf{V}_0^\dagger\mathbf{u}_{L,0} + \bar{\mathbf{d}}_{L,0}\mathbf{V}_0^\dagger\frac{\delta\Gamma}{\delta\bar{\mathbf{u}}_{L,0}}\right] &= 0 \\ \frac{\delta\Gamma}{\delta\phi_0^+}\,v_0 - \sqrt{2}\left[\frac{\delta\Gamma}{\delta\mathbf{u}_{L,0}}\mathbf{V}_0\mathbf{d}_{L,0} + \bar{\mathbf{u}}_{L,0}\mathbf{V}_0\frac{\delta\Gamma}{\delta\bar{\mathbf{d}}_{L,0}}\right] &= 0\,. \end{aligned} \tag{27}$$

The bare biunitary transformation (26) relating bare mass and elektroweak eigenstates is defined to diagonalize the

[1] Bare parameters and fields will be labeled with an additional subscript 0.

bare mass matrices. As a consequence in (27) the bare CKM matrix

$$\mathbf{V}_0 = \mathbf{S}_{u,0}^{L\,\dagger}\,\mathbf{S}_{d,0}^{L} \tag{28}$$

appears and the bare mass matrices of the mass eigenstates are diagonal. In other words, in order not to violate the Ward identities mass renormalization has to be performed in such a way that no off diagonal mass counterterms are needed.

Upon functional differentiation with respect to up- and down quark field sources the identities (27) relate quark matrix elements of the charged current (22) to the two point functions of the quarks. As a consequence the renormalization of the charge current which defines the renormalization prescription for the CKM matrix is completely determined by the wave function renormalization constants of the left handed quarks. This can be seen most easily as follows. If we assume that dimensional regularization respects the symmetry, the Ward identities are forminvariant under renormalization, i.e. they must hold also for the renormalized fields and parameters

$$\begin{aligned} \frac{\delta\Gamma}{\delta\phi^-}\,v + \sqrt{2}\left[\frac{\delta\Gamma}{\delta\mathbf{d}_L}\mathbf{V}^\dagger\mathbf{u}_L + \bar{\mathbf{d}}_L\mathbf{V}^\dagger\frac{\delta\Gamma}{\delta\bar{\mathbf{u}}_L}\right] &= 0 \\ \frac{\delta\Gamma}{\delta\phi_0^+}\,v - \sqrt{2}\left[\frac{\delta\Gamma}{\delta\mathbf{u}_L}\mathbf{V}\mathbf{d}_L + \bar{\mathbf{u}}_L\mathbf{V}\frac{\delta\Gamma}{\delta\bar{\mathbf{d}}_\mathbf{L}}\right] &= 0\,. \end{aligned} \tag{29}$$

For (27,29) both to be valid the bare and the renormalized CKM matrix have to fulfil the relations

$$\mathbf{V}_0 = \sqrt{\mathbf{Z}_{u,L}}\;\mathbf{V}\;\sqrt{\mathbf{Z}_{d,L}}^{\;-1} \tag{30}$$

$$\sqrt{\mathbf{Z}_{u,L}^\dagger\mathbf{Z}_{u,L}}\,\mathbf{V} = \mathbf{V}\,\sqrt{\mathbf{Z}_{d,L}^\dagger\mathbf{Z}_{d,L}} \tag{31}$$

where $\mathbf{V}_0$ is the bare, and $\mathbf{V}$ the renormalized CKM matrix and $\mathbf{Z}_{u/d,L}$ are the matrices of wave function renormalization of the left handed up and down quarks

$$\begin{aligned} \mathbf{u}_{L,0} &= \sqrt{\mathbf{Z}_{u,L}}\,\mathbf{u}_L \\ \mathbf{d}_{L,0} &= \sqrt{\mathbf{Z}_{d,L}}\,\mathbf{d}_L\,. \end{aligned} \tag{32}$$

Note that in perturbation theory we can evaluate the square root of these matrices as well as we can invert them. From equations (30, 31) it follows that the unitarity of the bare CKM matrix implies the unitarity of the renormalized CKM matrix (and *vice versa*) and has to be regarded as a constraint on the $\mathbf{Z}_{u/d,L}$. The renormalization group equation for the CKM matrix

$$\frac{d}{d\ln\mu}\mathbf{V} = \boldsymbol{\beta}_\mathbf{V} \tag{33}$$

is derived in the usual way by differentiating the bare CKM with respect to $\ln\mu$. Due to (30) this β-function can be expressed in terms of the anomalous dimension matrices of the fields

$$\boldsymbol{\gamma}_{u/d,L} = \mathbf{Z}_{u/d,L}^{-1}\frac{d}{d\ln\mu}\mathbf{Z}_{u/d,L} \tag{34}$$

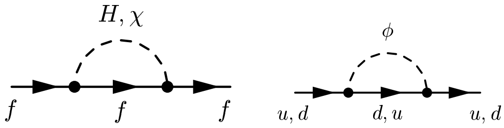

Fig. 1. Quark self energy diagrams

as

$$\boldsymbol{\beta}_\mathbf{V} = \frac{1}{2}[\mathbf{V}\boldsymbol{\gamma}_{d,L} - \boldsymbol{\gamma}_{u,L}\mathbf{V}]\,. \tag{35}$$

The second of the Ward identities allows us to eliminate the hermitian parts of the field anomalous dimensions and the final result for the β-function reads

$$\boldsymbol{\beta}_\mathbf{V} = \frac{1}{4}[\mathbf{V}(\boldsymbol{\gamma}_{d,L} - \boldsymbol{\gamma}_{d,L}^\dagger) - (\boldsymbol{\gamma}_{u,L} - \boldsymbol{\gamma}_{u,L}^\dagger)\mathbf{V}]\,. \tag{36}$$

The appearence of only the antihermitian part is natural since upon exponentiation, i.e. solving the renormalization group equation, this yields the unitary contribution to the field renormalization matrix which can be absorbed into a redefinition of the CKM matrix without destroying its unitarity. This relation still holds to all orders and to evaluate it at one loop one needs to compute the divergent part of the quark self energies shown in Fig. 1.

From this we obtain for the hermitian and antihermitian parts of the field anomalous dimensions for the left and right handed quarks

$$\left(\boldsymbol{\gamma}_{u,R}^{(1)} + \boldsymbol{\gamma}_{u,R}^{(1)\dagger}\right) = \frac{1}{4\pi^2v^2}\,2\mathbf{m}_u^2 \tag{37}$$

$$\left(\boldsymbol{\gamma}_{u,L}^{(1)} + \boldsymbol{\gamma}_{u,L}^{(1)\dagger}\right) = \frac{1}{4\pi^2v^2}\left[\mathbf{m}_u^2 + \mathbf{V}\mathbf{m}_d^2\mathbf{V}^\dagger\right] \tag{38}$$

$$\begin{aligned} \left(\boldsymbol{\gamma}_{u,R}^{(1)} - \boldsymbol{\gamma}_{u,R}^{(1)\dagger}\right)_{AC} &= \frac{1}{4\pi^2v^2}\,6\,\frac{m_{u,A}m_{u,C}}{m_{u,A}^2 - m_{u,C}^2} \\ &\quad\times\left(\mathbf{V}\mathbf{m}_d^2\mathbf{V}^\dagger\right)_{AC} \quad A \neq C \end{aligned} \tag{39}$$

$$\begin{aligned} \left(\boldsymbol{\gamma}_{u,L}^{(1)} - \boldsymbol{\gamma}_{u,L}^{(1)\dagger}\right)_{AC} &= \frac{1}{4\pi^2v^2}\,3\,\frac{m_{u,A}^2 + m_{u,C}^2}{m_{u,A}^2 - m_{u,C}^2} \\ &\quad\times\left(\mathbf{V}\mathbf{m}_d^2\mathbf{V}^\dagger\right)_{AC} \quad A \neq C \end{aligned} \tag{40}$$

$$\begin{aligned} \left(\boldsymbol{\gamma}_{u,R}^{(1)} - \boldsymbol{\gamma}_{u,R}^{(1)\dagger}\right)_{AA} &= \left(\boldsymbol{\gamma}_{u,L}^{(1)} - \boldsymbol{\gamma}_{u,L}^{(1)\dagger}\right)_{AA} \\ &= \frac{1}{4\pi^2v^2}\,3i\,s_{u,A} \end{aligned} \tag{41}$$

$$\left(\boldsymbol{\gamma}_{d,R}^{(1)} + \boldsymbol{\gamma}_{d,R}^{(1)\dagger}\right) = \frac{1}{4\pi^2v^2}\,2\mathbf{m}_d^2 \tag{42}$$

$$\left(\boldsymbol{\gamma}_{d,L}^{(1)} + \boldsymbol{\gamma}_{d,L}^{(1)\dagger}\right) = \frac{1}{4\pi^2v^2}\left[\mathbf{m}_d^2 + \mathbf{V}^\dagger\mathbf{m}_u^2\mathbf{V}\right] \tag{43}$$

$$\begin{aligned} \left(\boldsymbol{\gamma}_{d,R}^{(1)} - \boldsymbol{\gamma}_{d,R}^{(1)\dagger}\right)_{BD} &= \frac{1}{4\pi^2v^2}\,6\,\frac{m_{d,B}m_{d,D}}{m_{d,B}^2 - m_{d,D}^2} \\ &\quad\left(\mathbf{V}^\dagger\mathbf{m}_u^2\mathbf{V}\right)_{BD} \quad B \neq D \end{aligned} \tag{44}$$

$$\left(\gamma^{(1)}_{d,L} - \gamma^{(1)\dagger}_{d,L}\right)_{BD} = \frac{1}{4\pi^2 v^2}\, 3\, \frac{m^2_{d,B} + m^2_{d,D}}{m^2_{d,B} - m^2_{d,D}} \left(\mathbf{V}^\dagger \mathbf{m}^2_u \mathbf{V}\right)_{BD} \quad B \neq D \tag{45}$$

$$\left(\gamma^{(1)}_{d,R} - \gamma^{(1)\dagger}_{d,R}\right)_{BB} = \left(\gamma^{(1)}_{d,L} - \gamma^{(1)\dagger}_{d,L}\right)_{BB} = \frac{1}{4\pi^2 v^2}\, 3i\, s_{d,B} \tag{46}$$

where the capital letter indices run from 1 to 3, $m_{u/d,B}$ is the mass of the up/down type quark of the B'th family and the $s_{u/d,A}$ are explained after equation (50).

From the self energy diagrams in Fig. 1 we can also compute the Higgs contribution to the mass renormalization. The bare mass matrices are written as

$$\mathbf{m}^0_u = \mathbf{m}_u + \delta \mathbf{m}_u \tag{47}$$
$$\mathbf{m}^0_d = \mathbf{m}_d + \delta \mathbf{m}_d\,. \tag{48}$$

We choose the renormalization prescription such that the bare mass matrices and hence also $\delta\mathbf{m}_{u/d}$ are diagonal. This is possible, since we can absorb the off-diagonal elements into the off-diagonal elements of the antihermitian part of the right-handed wave function renormalization matrices according to

$$\begin{aligned}
&\not{p}\,\omega_+\, \boldsymbol{\Sigma}^{u,R}_{div} + \not{p}\,\omega_-\, \boldsymbol{\Sigma}^{u,L}_{div} + \mathbf{m}_u\, \omega_-\, \boldsymbol{\Sigma}^{u,S}_{div} + \boldsymbol{\Sigma}^{u,S}_{div}\, \mathbf{m}_u\, \omega_+ \\
&+\not{p}\,\omega_+ \frac{1}{2}\left(\delta\mathbf{Z}^{u,R} + \delta\mathbf{Z}^{u,R\,\dagger}\right) + \not{p}\,\omega_- \frac{1}{2}\left(\delta\mathbf{Z}^{u,L} + \delta\mathbf{Z}^{u,L\,\dagger}\right) \\
&-\frac{1}{2}\left(\mathbf{m}_u\, \delta\mathbf{Z}^{u,R} + \delta\mathbf{Z}^{u,L\,\dagger}\, \mathbf{m}_u\right)\omega_+ \\
&-\frac{1}{2}\left(\mathbf{m}_u\, \delta\mathbf{Z}^{u,L} + \delta\mathbf{Z}^{u,R\,\dagger}\, \mathbf{m}_u\right)\omega_- \\
&-\delta\mathbf{m}_u\, \omega_+ - \delta\mathbf{m}_u\, \omega_- = 0\,.
\end{aligned} \tag{49}$$

The $\boldsymbol{\Sigma}$ are given by a decomposition of the divergent parts of the unrenormalized self energy diagrams in Fig. 1

$$\begin{aligned}
\boldsymbol{\Sigma}^f_{div} &= \not{p}\,\omega_+\, \boldsymbol{\Sigma}^{f,R}_{div} + \not{p}\,\omega_-\, \boldsymbol{\Sigma}^{f,L}_{div} + \omega_+\, \mathbf{m}_f\, \boldsymbol{\Sigma}^{f,S}_{div} \\
&\quad + \boldsymbol{\Sigma}^{f,S}_{div}\, \mathbf{m}_f\, \omega_-\,.
\end{aligned} \tag{50}$$

The diagonal elements (41,46), i.e. the parameters $s_{u/d,A}$, are not fixed. This reflects the freedom to rephase the quark fields and from equation (49) it follows that the left and right handed contributions are identical but arbitrary.

Using (36) we derive the one loop β-function for the CKM matrix as

$$\begin{aligned}
\left(\boldsymbol{\beta}^{(1)}_{\mathbf{V}}\right)_{AB} = \frac{1}{16\pi^2 v^2}\, 3 \Bigg[& V_{AB}\, i\,(s_{d,B} - s_{u,A}) \\
& - \sum_{D\neq B}\sum_{C} V_{AD} V^*_{CD} V_{CB}\, \frac{m^2_{d,B} + m^2_{d,D}}{m^2_{d,B} - m^2_{d,D}}\, m^2_{u,C} \\
& - \sum_{C\neq A}\sum_{D} V_{AD} V^*_{CD} V_{CB}\, \frac{m^2_{u,A} + m^2_{u,C}}{m^2_{u,A} - m^2_{u,C}}\, m^2_{d,D} \Bigg]
\end{aligned} \tag{51}$$

which matches exactly the full SM result [3]. The one loop contribution of the Higgs interactions to the mass renormalization using our prescription is

$$\begin{aligned}
\delta m_{u,A} &= \frac{1}{16\pi^2 v^2}\, \Delta\, \frac{3}{2}\, m_{u,A} \left[m^2_{u,A} - \left(\mathbf{V}\mathbf{m}^2_d \mathbf{V}^\dagger\right)_{AA} \right] \\
\delta m_{d,B} &= \frac{1}{16\pi^2 v^2}\, \Delta\, \frac{3}{2}\, m_{d,B} \left[m^2_{d,B} - \left(\mathbf{V}^\dagger \mathbf{m}^2_u \mathbf{V}\right)_{BB} \right]
\end{aligned} \tag{52}$$

where $\Delta = 2/\varepsilon$. Finally also the wave function renormalization constant Z_H of the Higgs field is needed since this governs the renormalization of the vacuum expectation value

$$v_0 = Z_H\, v\,. \tag{53}$$

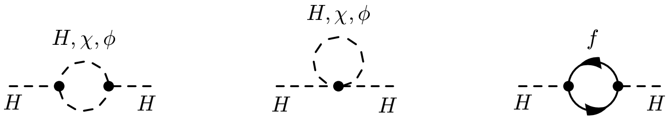

Fig. 2. Self energy of the physical Higgs

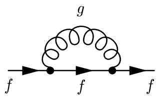

Fig. 3. QCD contribution to the quark self energy

From the Higgs self energy diagrams shown in Fig. 2 we extract the one loop contribution to the Higgs field renormalization constant

$$\delta Z_H = -\frac{1}{16\pi^2 v^2}\, \Delta\, 2 N_c\, \mathrm{Tr}\left(\mathbf{m}^2_u + \mathbf{m}^2_d\right)\,. \tag{54}$$

As far as the elektroweak interaction is concerned this will also be the full answer for $\boldsymbol{\beta}^{(1)}_{\mathbf{V}}$ since we are working at one loop. However, compared to the elektroweak contribution a much larger effect is the renormalization due to the strong interactions which do not induce flavor mixing and thus modify only the renormalization group functions for the masses but not the β-function for the CKM matrix. Computing the QCD self energies shown in Fig. 3 it turns out that we have

$$\begin{aligned}
\delta m_{u,A} &= \frac{1}{16\pi^2 v^2}\, \Delta\, \frac{3}{2}\, m_{u,A} \left[m^2_{u,A} - \left(\mathbf{V}\mathbf{m}^2_d \mathbf{V}^\dagger\right)_{AA} \right] \\
&\quad -2\frac{\alpha_s}{\pi}\frac{\Delta}{2}\, m_{u,A} \\
\delta m_{d,B} &= \frac{1}{16\pi^2 v^2}\, \Delta\, \frac{3}{2}\, m_{d,B} \left[m^2_{d,B} - \left(\mathbf{V}^\dagger \mathbf{m}^2_u \mathbf{V}\right)_{BB} \right] \\
&\quad -2\frac{\alpha_s}{\pi}\frac{\Delta}{2}\, m_{d,A}\,.
\end{aligned} \tag{55}$$

Thus we have gathered all the one loop contributions needed to perform a renormalization group study of the CKM matrix.

$$\mathbf{V} = \begin{pmatrix} \mathrm{c}_{12}\,\mathrm{c}_{13} & \mathrm{s}_{12}\,\mathrm{c}_{13} & \mathrm{s}_{13}\,e^{-i\delta_{13}} \\ -\mathrm{s}_{12}\,\mathrm{c}_{23} - \mathrm{c}_{12}\,\mathrm{s}_{23}\,\mathrm{s}_{13}\,e^{i\delta_{13}} & \mathrm{c}_{12}\,\mathrm{c}_{23} - \mathrm{s}_{12}\,\mathrm{s}_{23}\,\mathrm{s}_{13}\,e^{i\delta_{13}} & \mathrm{s}_{23}\,\mathrm{c}_{13} \\ \mathrm{s}_{12}\,\mathrm{s}_{23} - \mathrm{c}_{12}\,\mathrm{c}_{23}\,\mathrm{s}_{13}\,e^{i\delta_{13}} & -\mathrm{c}_{12}\,\mathrm{s}_{23} - \mathrm{s}_{12}\,\mathrm{c}_{23}\,\mathrm{s}_{13}\,e^{i\delta_{13}} & \mathrm{c}_{23}\,\mathrm{c}_{13} \end{pmatrix} \tag{67}$$

4 Renormalization group flow

We already derived the β-function for the CKM matrix and it only remains to obtain the mass anomalous dimensions from (55)

$$\gamma_{u,A} = -m_{u,A}^{-1}\,\mu\frac{d}{d\mu}\,\delta m_{u,A}$$
$$\gamma_{d,B} = -m_{d,B}^{-1}\,\mu\frac{d}{d\mu}\,\delta m_{d,B}. \tag{56}$$

and the anomalous dimension γ_v of the vacuum expectation value from (54)

$$\gamma_v^{(1)} = \frac{\varepsilon}{2} - \mu\frac{d}{d\mu}\delta Z_H \tag{57}$$

in order to end up with a closed set of differential equations. The term linear in ε appears in (57) since the vacuum expectation value changes its dimensionality in dimensional regularization. The total derivative with respect to $\ln\mu$ is in general

$$\begin{aligned}\mu\frac{d}{d\mu} &= \mu\frac{\partial}{\partial\mu} + v\gamma_v\frac{\partial}{\partial v} + M_H^2\gamma_H\frac{\partial}{\partial M_H^2} \\ &+ M_\chi^2\gamma_\chi\frac{\partial}{\partial M_\chi^2} + \mathbf{m}_u\boldsymbol{\gamma}_u\frac{\partial}{\partial\mathbf{m}_u} \\ &+ \mathbf{m}_d\boldsymbol{\gamma}_d\frac{\partial}{\partial\mathbf{m}_d} + \boldsymbol{\beta}_{\mathbf{V}}\frac{\partial}{\partial\mathbf{V}} + \beta_{\alpha_s}\frac{\partial}{\partial\alpha_s}\end{aligned} \tag{58}$$

but to one loop this reduces to

$$\mu\frac{d}{d\mu} = \frac{1}{\Delta}v\frac{\partial}{\partial v} - 2\alpha_s\frac{1}{\Delta}\frac{\partial}{\partial\alpha_s}\,. \tag{59}$$

Since the various renormalization group functions are obtained by acting with the total derivative on one loop contributions to the bare parameters and renormalization constants, we have kept in (59) only the lowest order contributions, i.e. the terms linear in ε. These cancel with the divergent parts of the one loop terms in the bare quantities yielding a finite contribution to the one loop renormalization group functions. In this way we obtain for the mass anomalous dimensions

$$\gamma_{u,A}^{(1)} = \frac{1}{16\pi^2v^2}\,3\left[m_{u,A}^2 - \left(\mathbf{V}\mathbf{m}_d^2\mathbf{V}^\dagger\right)_{AA}\right] - 2\frac{\alpha_s}{\pi}$$
$$\gamma_{d,B}^{(1)} = \frac{1}{16\pi^2v^2}\,3\left[m_{d,B}^2 - \left(\mathbf{V}^\dagger\mathbf{m}_u^2\mathbf{V}\right)_{BB}\right] - 2\frac{\alpha_s}{\pi} \tag{60}$$

and for the anomalous dimension of the vacuum expectation value

$$\gamma_v^{(1)} = -\frac{1}{16\pi^2v^2}\,2N_c\,\mathrm{Tr}\left(\mathbf{m}_u^2 + \mathbf{m}_d^2\right) \tag{61}$$

where we have dropped the term linear in ε.

Thus the complete set of differential equations is

$$\mu\frac{d}{d\mu}\,v = \gamma_v^{(1)}\,v \tag{62}$$
$$\mu\frac{d}{d\mu}\,m_{u,A} = \gamma_{u,A}^{(1)}\,m_{u,A} \tag{63}$$
$$\mu\frac{d}{d\mu}\,m_{d,B} = \gamma_{d,B}^{(1)}\,m_{d,B} \tag{64}$$
$$\mu\frac{d}{d\mu}V_{AB} = \left(\boldsymbol{\beta}_{\mathbf{V}}^{(1)}\right)_{AB} \tag{65}$$
$$\mu\frac{d}{d\mu}\,\alpha_s = -2\alpha_s\frac{\alpha_s}{\pi}\beta^{(1)} \tag{66}$$

where $\beta^{(1)} = (33 - 2n_f)/12$. This set of equations is still valid for an arbitrary number of families. The case of two families is practical trivial and hence we switch directly to the relevant case of three families. Instead of working with the full matrices V_{AB} we choose the standard parametrization of the Particle Data Group [10] (see (67) on top of the page) with

$$\begin{aligned}&\mathrm{s}_{12} = \sin(\theta_{12}),\ \mathrm{c}_{12} = \cos(\theta_{12}),\\ &\mathrm{s}_{13} = \sin(\theta_{13}),\ \mathrm{c}_{13} = \cos(\theta_{13}),\\ &\mathrm{s}_{23} = \sin(\theta_{23}),\ \mathrm{c}_{23} = \cos(\theta_{23})\end{aligned} \tag{68}$$

and write the renormalization group equations for the three angles θ_{ij} and the phase δ_{13}

$$\begin{aligned}\beta_{12}^{(1)} &= \mu\frac{d}{d\mu}\,\theta_{12},\\ \beta_{23}^{(1)} &= \mu\frac{d}{d\mu}\,\theta_{23},\\ \beta_{13}^{(1)} &= \mu\frac{d}{d\mu}\,\theta_{13},\\ \beta_{\delta}^{(1)} &= \mu\frac{d}{d\mu}\,\delta_{13}.\end{aligned} \tag{69}$$

The expressions for $\beta_{ij}^{(1)}$, $\beta_\delta^{(1)}$ and $\gamma_{u/d,A}^{(1)}$ in terms of the angles θ_{ij} and the phase δ_{13} are quite lengthy and are deferred to the appendix. Together with the equations for the masses (63, 64), the vacuum expectation value (62) and the strong coupling constant (66) this is a coupled system of 12 differential equations which cannot be solved analytically without approximations. We haved studied the exact solutions numerically and found that they are reproduced with excellent accuracy by an approximative analytical solution to be discussed below.

We shall study the renormalization group flow starting at the scale of top-quark mass; the discussion of the renormalization group flow below this scale is a separate issue since on then has to integrate out the top, bottom and charm quarks at the appropriate mass scales.

As initial values at the scale $\mu_0 \approx m_t$ we choose [10, 11]

$$
\begin{aligned}
&v = 245.3\ \text{GeV}, && \alpha_s = 0.109, \\
&\theta_{12} = 0.221, && \theta_{23} = 0.039, \\
&\theta_{13} = 0.0031, && \delta_{13} = 1.26, \\
&m_{u,3} \equiv m_t = 165.8\ \text{GeV}. &&
\end{aligned}
\tag{70}
$$

The initial value of the running top mass $m_t(\mu)$ in the $\overline{MS}$-scheme is related to the pole mass in the usual way

$$
m_t(m_t^{pole}) = m_t^{pole}\,[1 - \frac{\alpha_s}{\pi} C_F]\,, \tag{71}
$$

where $C_F = 4/3$ and the top quark pole mass is given by the experimental measured value $m_t^{pole} = (173.8 \pm 5.2)$ GeV.

The masses of the light quarks (u, c, d, s, b) at m_t have been evolved from low scale by QCD corrections only:

$$
\begin{aligned}
m_{u,1} &\equiv m_u = 2.0\ \text{MeV}, \\
m_{d,1} &\equiv m_d = 3.7\ \text{MeV}, \\
m_{u,2} &\equiv m_c = 0.72\ \text{GeV}, \\
m_{d,2} &\equiv m_s = 72\ \text{MeV}, \\
m_{d,3} &\equiv m_b = 3.0\ \text{GeV}.
\end{aligned}
\tag{72}
$$

Note that our results for the CKM matrix elements do not depend critically on the exact values for the five light quark masses, hence we do not need to include uncertainties in these masses.

The observed mass spectrum of the quarks together with the hierarchy of the CKM angles allows us to construct an excellent approximation for this system which can be solved analytically.

First of all we observe that the ratios $(m_{u/d,A}^2 + m_{u/d,C}^2)/(m_{u/d,A}^2 - m_{u/d,C})^2$ appearing in β_{ij} and β_δ are due to the large differences in the quark masses practically ± 1. Furthermore the renormalization group functions depend on $m_{u/d,A}^2/v^2$ which is extremely small except for the top quark. Hence we neglect these terms and obtain

$$
\begin{aligned}
\beta_{12} &= \frac{3}{16\pi^2}\, c_{12} \Big[s_{12}\, \{ s_{23}^2 - c_{23}^2\, s_{13}^2 \} \\
&\qquad - 2 c_{12}\, c_{23}\, s_{23}\, s_{13}\, \cos(\delta_{13}) \Big] \frac{m_t^2}{v^2} \\
\beta_{23} &= \frac{3}{16\pi^2}\, c_{23}\, s_{23}\, \frac{m_t^2}{v^2} \\
\beta_{13} &= \frac{3}{16\pi^2}\, c_{23}^2\, c_{13}\, s_{13}\, \frac{m_t^2}{v^2} \\
\beta_\delta &= \frac{3}{16\pi^2}\, \frac{c_{12}\, c_{23}\, s_{23}\, s_{13}}{s_{12}}\, \sin(\delta_{13})\, \frac{m_t^2}{v^2} \\
\gamma_v &= -\frac{6}{16\pi^2}\, \frac{m_t^2}{v^2} \\
\gamma_{u,1} &= -2\frac{\alpha_s}{\pi} \\
\gamma_{u,2} &= -2\frac{\alpha_s}{\pi} \\
\gamma_{u,3} &= \frac{3}{16\pi^2}\, \frac{m_t^2}{v^2} - 2\frac{\alpha_s}{\pi} \\
\gamma_{d,1} &= -\frac{3}{16\pi^2} \Big[s_{12}^2\, s_{23}^2 + c_{12}^2\, c_{23}^2\, s_{13}^2 \\
&\qquad - 2\, c_{12}\, c_{23}\, s_{12}\, s_{23}\, s_{13}\, \cos(\delta_{13}) \Big] \frac{m_t^2}{v^2} \\
&\quad - 2\frac{\alpha_s}{\pi} \\
\gamma_{d,2} &= -\frac{3}{16\pi^2} \Big[c_{12}^2\, s_{23}^2 + s_{12}^2\, c_{23}^2\, s_{13}^2 \\
&\qquad + 2\, c_{12}\, c_{23}\, s_{12}\, s_{23}\, s_{13}\, \cos(\delta_{13}) \Big] \frac{m_t^2}{v^2} \\
&\quad - 2\frac{\alpha_s}{\pi} \\
\gamma_{d,3} &= -\frac{3}{16\pi^2}\, c_{23}^2\, c_{13}^2\, \frac{m_t^2}{v^2} - 2\frac{\alpha_s}{\pi}\,.
\end{aligned}
\tag{73}
$$

Secondly we make use of the hierarchy of the CKM angles

$$
\theta_{12} = \mathcal{O}(10^{-1}), \quad \theta_{23} = \mathcal{O}(10^{-2}), \quad \theta_{13} = \mathcal{O}(10^{-3}) \tag{74}
$$

thus keeping only terms of $\mathcal{O}(10^{-3})$ in $\gamma_{u/d,A}$, β_{ij}/θ_{ij} and β_δ/δ. From this we obtain the very simple system

$$
\begin{aligned}
\beta_{23} &= \frac{3}{16\pi^2}\, \frac{m_t^2}{v^2}\, \theta_{23} \\
\beta_{13} &= \frac{3}{16\pi^2}\, \frac{m_t^2}{v^2}\, \theta_{13} \\
\gamma_{d,1} &= -2\frac{\alpha_s}{\pi} \\
\gamma_{d,2} &= -2\frac{\alpha_s}{\pi} \\
\gamma_{d,3} &= -\frac{3}{16\pi^2}\, \frac{m_t^2}{v^2} - 2\frac{\alpha_s}{\pi} \\
\gamma_{u,1} &= -2\frac{\alpha_s}{\pi} \\
\gamma_{u,2} &= -2\frac{\alpha_s}{\pi} \\
\gamma_{u,3} &= \frac{3}{16\pi^2}\, \frac{m_t^2}{v^2} - 2\frac{\alpha_s}{\pi} \\
\gamma_v &= -2\, \frac{3}{16\pi^2}\, \frac{m_t^2}{v^2}\,.
\end{aligned}
\tag{75}
$$

The right hand side is determined by the top-Yukawa coupling $Y_t = m_t/v$ for which we derive the renormalization group equation

$$
\frac{d}{\ln\mu}\, Y_t = Y_t \left[\frac{9}{16\pi^2}\, Y_t^2 - 2\frac{\alpha_s(\mu)}{\pi} \right] \tag{76}
$$

which can be solved analytically

$$Y_t(\mu) = \left[Y_t^{-2}(\mu_0)\left(\frac{\alpha_s(\mu_0)}{\alpha_s(\mu)}\right)^{\frac{2}{\beta^{(1)}}} - \frac{9}{16\pi}\,\frac{1}{\beta^{(1)}-2}\,\frac{1}{\alpha_s(\mu)} \times\left\{1-\left(\frac{\alpha_s(\mu_0)}{\alpha_s(\mu)}\right)^{\frac{2}{\beta^{(1)}}-1}\right\}\right]^{-\frac{1}{2}}. \quad (77)$$

Where $\beta^{(1)}$ is the one-loop QCD β-function given after equation (66). In leading logarithmic approximation the running of the strong coupling constant is given by the solution of (66)

$$\alpha_s(\mu) = \frac{\alpha_s(\mu_0)}{1+\frac{2}{\pi}\,\beta^{(1)}\,\alpha_s(\mu_0)\ln\frac{\mu}{\mu_0}}. \quad (78)$$

The differential equations (75) for the masses, angles and the vacuum expectation value can be written in the compact form

$$\frac{1}{y}\,\mu\,\frac{d}{d\mu}\,y = \frac{3}{16\pi^2}\,c_y\,Y_t^2\,(\mu) - q_y\,2\frac{\alpha_s(\mu)}{\pi} \quad (79)$$

where $y = \theta_{12}, \theta_{13}, \theta_{23}, \delta_{13}, m_u, m_d, m_c, m_s, m_b, m_t, v$ and

$$c_y = \begin{cases} 1 \text{ if } y = \theta_{23}, \theta_{13}, m_t \\ -1 \text{ if } y = m_b \\ -2 \text{ if } y = v \\ 0 \text{ if } y = \theta_{12}, \delta_{13}, m_u, m_d, m_s, m_c \end{cases} \quad (80)$$

and

$$q_y = \begin{cases} 0 \text{ if } y = \theta_{12}, \theta_{23}, \theta_{13}, \delta_{13}, v \\ 1 \text{ if } y = m_u, m_d, m_s, m_c, m_b, m_t. \end{cases} \quad (81)$$

The analytical solution of (79) reads

$$y(\mu) = y(\mu_0)\left[1+\frac{9}{16\pi}\,\frac{1}{\beta^{(1)}-2}\,\frac{1}{\alpha_s(\mu_0)}\,\frac{m_t^2(\mu_0)}{v^2(\mu_0)} \times\left\{1-\left(\frac{\alpha_s(\mu_0)}{\alpha_s(\mu)}\right)^{1-\frac{2}{\beta^{(1)}}}\right\}\right]^{-\frac{1}{6}c_y} \times\left[\frac{\alpha_s(\mu_0)}{\alpha_s(\mu)}\right]^{-\frac{1}{\beta^{(1)}}q_y} \quad (82)$$

and in particular for the CKM matrix elements $V_{ub} \approx \theta_{13}e^{i\delta_{13}}$ and $V_{cb} \approx \theta_{23}$

$$\frac{|V_{ub}(\mu)|}{|V_{ub}(\mu_0)|} = \frac{|V_{cb}(\mu)|}{|V_{cb}(\mu_0)|} = \left[1+\frac{9}{16\pi}\,\frac{1}{\beta^{(1)}-2}\,\frac{1}{\alpha_s(\mu_0)}\,\frac{m_t^2(\mu_0)}{v^2(\mu_0)} \times\left\{1-\left(\frac{\alpha_s(\mu_0)}{\alpha_s(\mu)}\right)^{1-\frac{2}{\beta^{(1)}}}\right\}\right]^{-\frac{1}{6}}. \quad (83)$$

As a cross check we also solved the equations (62) to (66) numerically without any approximation. The deviation of the analytic solution from these numerical results are estimated by the size of the integration interval of $\ln\mu$ of order $\mathcal{O}(10)$ times the size of the terms of order $\mathcal{O}(10^{-4})$ neglected in the RG- and β-functions. Indeed the deviation is less than a half percent for each parameter.

The results of the analytic solution are shown in the Figs. 4 to 7 in the next chapter together with the results of the full SM.

5 Comparison with the full SM

Up to now we have neglected the complete electroweak part and all leptons of the SM. To be able to test the precission of our analytic approximation we have solved the renormalization group equations numerically in the full SM. This approach has also been chosen in [4–8].

Five additional parameters are entering the analysis. A convenient choice is the coupling constant g_Y of the $U(1)_Y$-hypercharge, the coupling constant g_W of the $SU(2)_L$-weak interaction and the masses of the electron, the muon and the tau. The one-loop results for the RG-functions of these parameters are

$$\gamma_{g_Y^2} = \frac{g_Y^2}{4\pi^2}\,\frac{1}{12}\left(1+\sum_{fermions}(Y_L^2+Y_R^2)\right) \quad (84)$$

$$\gamma_{g_W^2} = \frac{g_W^2}{4\pi^2}\,\frac{-43+2\,N_m}{12} \quad (85)$$

$$\gamma_l = \frac{g_Y^2}{4\pi^2}\,\frac{-11}{16}+\frac{g_W^2}{4\pi^2}\,\frac{3}{16}+\frac{3m_l^2}{16\pi^2 v^2} \quad (l = e, \mu, \tau) \quad (86)$$

where N_m is the number of $SU(2)_L$ multiplets. Furthermore, in the RG-functions of the quarks given in appendix an additional term appears

$$\gamma_{u,A}^{elweak} = \frac{g_Y^2}{4\pi^2}\,\frac{-5}{48}+\frac{g_W^2}{4\pi^2}\,\frac{3}{16} \qquad (A = u, c, t) \quad (87)$$

$$\gamma_{d,B}^{elweak} = \frac{g_Y^2}{4\pi^2}\,\frac{7}{48}+\frac{g_W^2}{4\pi^2}\,\frac{3}{16} \qquad (B = d, s, b) \quad (88)$$

while for the vacuum expectation value the term

$$\gamma_v^{elweak} = \frac{g_Y^2}{4\pi^2}\,\frac{1}{4}+\frac{g_W^2}{4\pi^2}\,\frac{3}{4} \quad (89)$$

must be added. As already pointed out in Sect. 3 the β-functions for the CKM parameters in the full SM are the same as in the "ungauged" model. This follows from the fact that the electroweak contributions to the divergent part of Σ^R and Σ^L are diagonal, flavor independent and real. In other words, they do not have any antihermitian parts and thus cannot contribute to the renormalization constant of the CKM matrix.

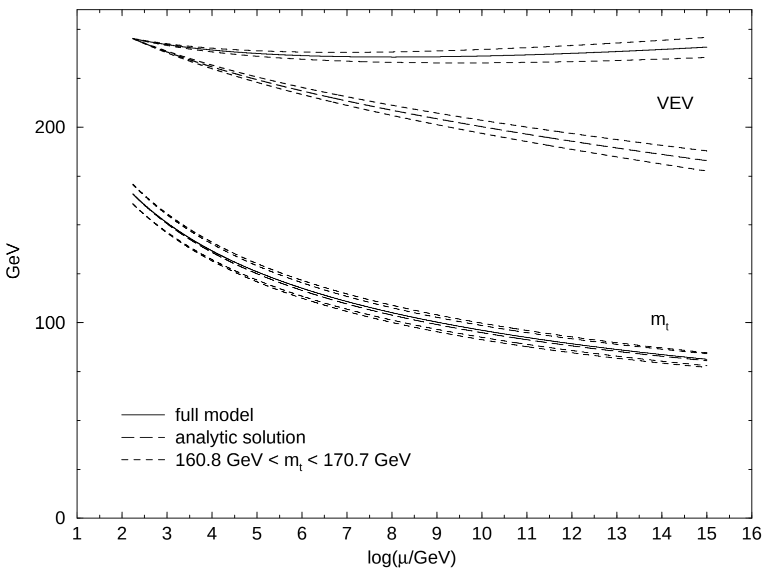

Fig. 4. Renormalization group scaling of m_t und v. The width of the bands reflects the uncertainty of $\pm 5.2\,\mathrm{GeV}$ in the top quark pole mass

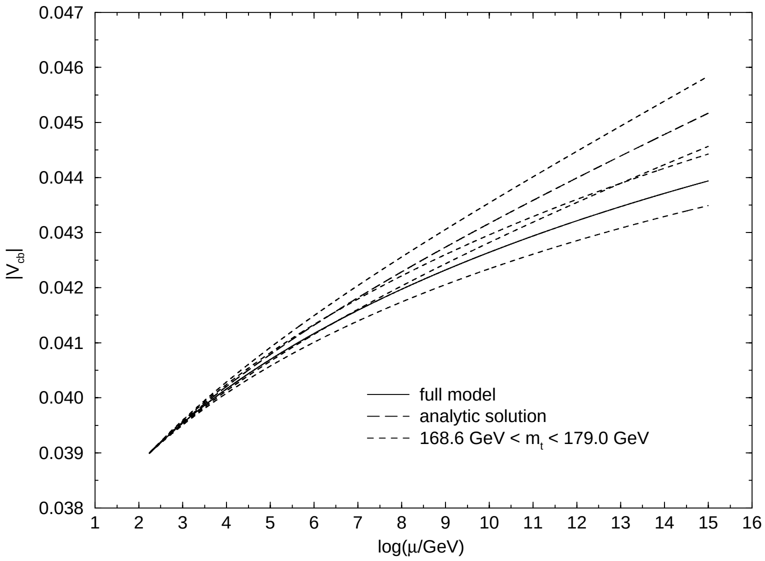

Fig. 5. Renormalization group evolution of V_{cb}

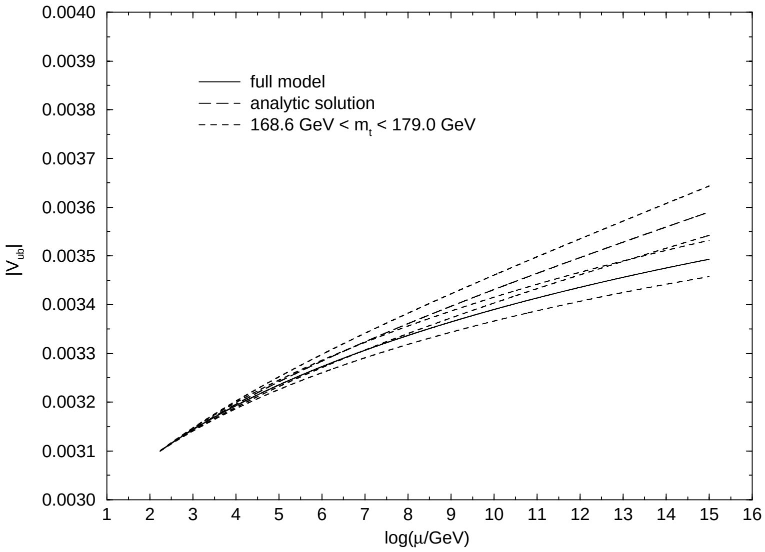

Fig. 6. Renormalization group evolution of V_{ub}

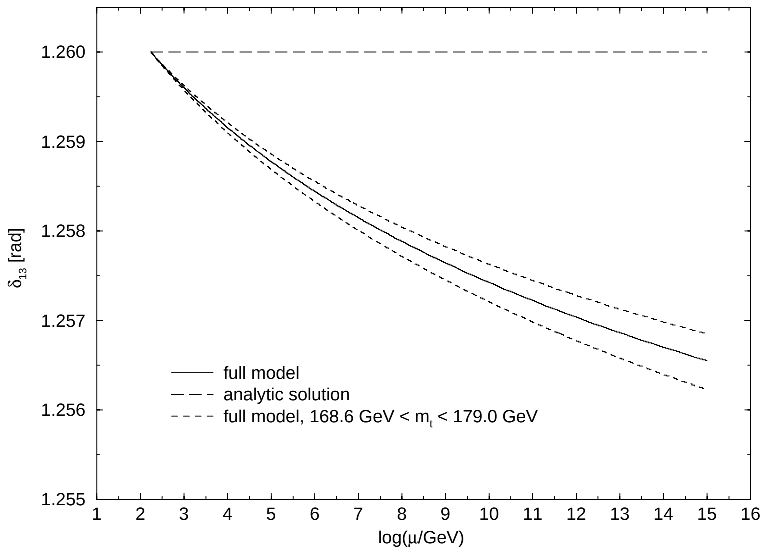

Fig. 7. Renormalization group evolution of the phase δ_{13}

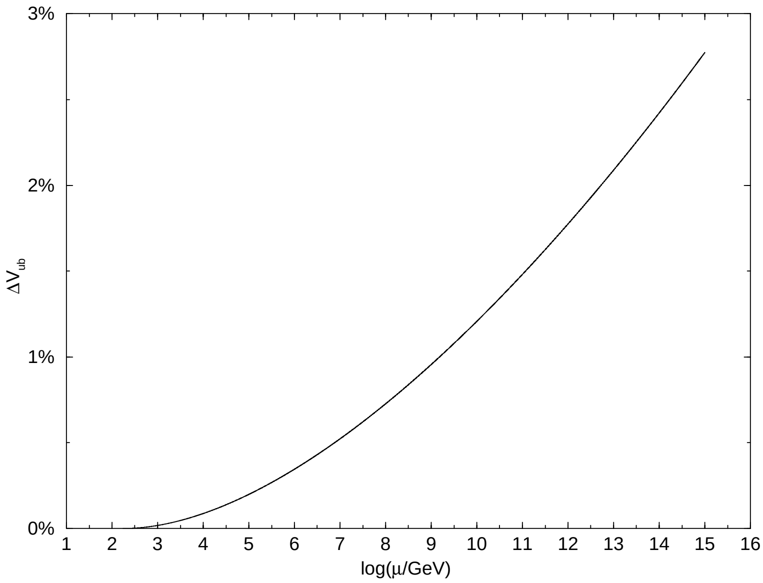

Fig. 8. The relative difference between the analytic solution and the numerical results in the full SM for V_{ub}

The RG-functions for the coupling constants decouple and lead to:

$$\alpha_Y(\mu) = \frac{g_Y^2}{4\pi} = \frac{1}{\frac{1}{\alpha_Y(\mu_0)} - \frac{41}{12\pi}\ln(\frac{\mu}{\mu_0})} \quad (\mu, \mu_0 \geq m_t) \qquad (90)$$

$$\alpha_W(\mu) = \frac{g_W^2}{4\pi} = \frac{1}{\frac{1}{\alpha_W(\mu_0)} + \frac{19}{12\pi}\ln(\frac{\mu}{\mu_0})} \quad (\mu, \mu_0 \geq m_t). \qquad (91)$$

Figure 4 shows the running of the top quark mass and the vacuum expectation value between m_t and the large scale 10^{15} GeV according to our analytical approximation and the numerical results in the full SM. While the vacuum expectation value differs significantly, the top quark mass is practically the same in both cases.

The renormalization group evolution of the CKM matrix elements $|V_{cb}|$ and $|V_{ub}|$ is plotted in Figs. 5 and 6. Figure 7 shows that the phase δ_{13} is practically constant up to the GUT scale even in the full SM.

Although the difference between the analytic solution and the full SM is significant for the vacuum expectation value, the relative deviations for the CKM matrix elements are less than 3% for $|V_{cb}|$ and $|V_{ub}|$ in the whole range of m_t up to the GUT scale. In our approximation these elements are identical with the parameters θ_{23} and θ_{13}. The approximation of constant parameters θ_{12} and δ_{13} is even better with a relative difference of less than a half percent. As an example the relative deviation of $|V_{ub}|$ is plotted in Fig. 8. The corresponding plot for V_{cb} is the same within the width of the lines.

6 Conclusions

We have studied the scale dependence of the CKM parameters and quark masses. To one loop order it turned out that an "ungauged" SM (i.e. only the Higgs) sector yields the same one loop result for the β-function for the CKM matrix as the full $SU(3)_C \otimes SU(2)_L \otimes U(1)_Y$ gauge theory. This is obvious, since the renormalization of the Yukawa couplings due to transverse gauge bosons is proportional to the unit matrix and hence does not change $\delta\mathbf{V}$. The renormalization of the CKM matrix is governed by a Ward identity which allows us to express its renormalization solely in terms of the quark self energies. This result is valid in the full SM and we exploited it in the Higgs sector to derive the renormalization group functions for the masses and CKM parameters.

Studying only the Higgs sector it becomes obvious that the running of the CKM matrix is governed by the Yukawa couplings which are very small except for the top quark. This motivates the limit in which all quark masses except the one of the top are set to zero. However, it is well known that in such a limit no mixing can occure due to the degeneracy of the down type quark masses. In our results this is reflected by the appearence of the ratios $(m^2_{u/d,A} + m^2_{u/d,C})/(m^2_{u/d,A} - m^2_{u/d,C})$. The limiting values of these single out a basis in flavor space relative to which the CKM rotation can be defined. Using the physical values of the masses this ratios are either $+1$ or -1 which simplifies the renormalization group equations significantly. Putting in also the hierarchy of the CKM angles the renormalization group equations can be solved analytically with an accuracy better than a few percent.

The renormalization group flow of the mixing between the first two families turns out to be very small since the corresponding Yukawa couplings are tiny.

For the third family the effects become sizeable and the parameters such as V_{cb} and V_{ub} change at a level of 16% between m_t and the large scale 10^{15} GeV. In the full SM this increase is reduced to 13%. However, for the CKM matrix elements the analytic approximation solution differs less than 3% from the full numerical results. Since in our approximation V_{us} and the ratio V_{ub}/V_{cb} do not change with the renormalization scale all Wolfenstein parameters [12] except $A = V_{cb}/V^2_{us}$ are scale independent.

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Appendix: RG- and β–functions in the particle data group parametrization

$$
\begin{aligned}
\beta_{12} = \frac{3}{16\pi^2 v^2} \Bigg\{ & \frac{m_{d,1}^2 + m_{d,2}^2}{m_{d,1}^2 - m_{d,2}^2} \Bigg[m_{u,1}^2 \, \mathrm{s}_{12}\, \mathrm{c}_{12}\, \mathrm{c}_{13}^2 \\
& + m_{u,2}^2 \Bigg(\Big\{ \mathrm{s}_{12}^2 - \mathrm{c}_{12}^2 \Big\} \mathrm{s}_{23}\, \mathrm{c}_{23}\, \mathrm{s}_{13}\, \cos(\delta_{13}) + \mathrm{s}_{12}\, \mathrm{c}_{12} \Big\{ \mathrm{s}_{23}^2\, \mathrm{s}_{13}^2 - \mathrm{c}_{23}^2 \Big\} \Bigg) \\
& + m_{u,3}^2 \Bigg(- \Big\{ \mathrm{s}_{12}^2 - \mathrm{c}_{12}^2 \Big\} \mathrm{s}_{23}\, \mathrm{c}_{23}\, \mathrm{s}_{13}\, \cos(\delta_{13}) - \mathrm{s}_{12}\, \mathrm{c}_{12} \Big\{ \mathrm{s}_{23}^2 - \mathrm{c}_{23}^2\, \mathrm{s}_{13}^2 \Big\} \Bigg) \Bigg] \\
& + \frac{m_{d,1}^2 + m_{d,3}^2}{m_{d,1}^2 - m_{d,3}^2} \Bigg[m_{u,1}^2 \, \mathrm{s}_{12}\, \mathrm{c}_{12}\, \mathrm{s}_{13}^2 \\
& - m_{u,2}^2 \, \mathrm{s}_{12} \mathrm{s}_{23}\, \mathrm{s}_{13} \Big(\mathrm{c}_{12}\, \mathrm{s}_{23}\, \mathrm{s}_{13} + \mathrm{s}_{12}\, \mathrm{c}_{23}\, \cos(\delta_{13}) \Big) \\
& - m_{u,3}^2 \, \mathrm{s}_{12} \mathrm{c}_{23}\, \mathrm{s}_{13} \Big(\mathrm{c}_{12}\, \mathrm{c}_{23}\, \mathrm{s}_{13} - \mathrm{s}_{12}\, \mathrm{s}_{23}\, \cos(\delta_{13}) \Big) \Bigg] \\
& + \frac{m_{d,2}^2 + m_{d,3}^2}{m_{d,2}^2 - m_{d,3}^2} \Bigg[- m_{u,1}^2 \, \mathrm{s}_{12}\, \mathrm{c}_{12}\, \mathrm{s}_{13}^2 \\
& + m_{u,2}^2 \, \mathrm{c}_{12}\, \mathrm{s}_{23}\, \mathrm{s}_{13} \Big(\mathrm{s}_{12}\, \mathrm{s}_{23}\, \mathrm{s}_{13} - \mathrm{c}_{12}\, \mathrm{c}_{23}\, \cos(\delta_{13}) \Big) \\
& + m_{u,3}^2 \, \mathrm{c}_{12}\, \mathrm{c}_{23}\, \mathrm{s}_{13} \Big(\mathrm{s}_{12}\, \mathrm{c}_{23}\, \mathrm{s}_{13} + \mathrm{c}_{12}\, \mathrm{s}_{23}\, \cos(\delta_{13}) \Big) \Bigg] \\
& + \frac{m_{u,1}^2 + m_{u,2}^2}{m_{u,1}^2 - m_{u,2}^2} \Bigg[m_{d,1}^2 \, \mathrm{c}_{12}\, \mathrm{c}_{23} \Big(\mathrm{s}_{12}\, \mathrm{c}_{23} + \mathrm{c}_{12}\, \mathrm{s}_{23}\, \mathrm{s}_{13}\, \cos(\delta_{13}) \Big) \\
& - m_{d,2}^2 \, \mathrm{s}_{12}\, \mathrm{c}_{23} \Big(\mathrm{c}_{12}\, \mathrm{c}_{23} - \mathrm{s}_{12}\, \mathrm{s}_{23}\, \mathrm{s}_{13}\, \cos(\delta_{13}) \Big) - m_{d,3}^2 \, \mathrm{s}_{23}\, \mathrm{c}_{23}\, \mathrm{s}_{13}\, \cos(\delta_{13}) \Bigg] \\
& + \frac{m_{u,1}^2 + m_{u,3}^2}{m_{u,1}^2 - m_{u,3}^2} \Bigg[m_{d,1}^2 \, \mathrm{c}_{12}\, \mathrm{s}_{23} \Big(\mathrm{s}_{12}\, \mathrm{s}_{23} - \mathrm{c}_{12}\, \mathrm{c}_{23}\, \mathrm{s}_{13}\, \cos(\delta_{13}) \Big) \\
& - m_{d,2}^2 \, \mathrm{s}_{12} \mathrm{s}_{23} \Big(\mathrm{c}_{12}\, \mathrm{s}_{23} + \mathrm{s}_{12}\, \mathrm{c}_{23}\, \mathrm{s}_{13}\, \cos(\delta_{13}) \Big) + m_{d,3}^2 \, \mathrm{s}_{23}\, \mathrm{c}_{23}\, \mathrm{s}_{13}\, \cos(\delta_{13}) \Bigg] \Bigg\}
\end{aligned}
\tag{92}
$$

$$
\begin{aligned}
\beta_{23} = \frac{3}{16\pi^2 v^2} \Bigg\{ & \frac{m_{d,1}^2 + m_{d,3}^2}{m_{d,1}^2 - m_{d,3}^2} \Bigg[- m_{u,1}^2 \, \mathrm{s}_{12}\, \mathrm{c}_{12}\, \mathrm{s}_{13}\, \cos(\delta_{13}) \\
& + m_{u,2}^2 \, \mathrm{s}_{12}\, \mathrm{s}_{23} \Big(\mathrm{s}_{12}\, \mathrm{c}_{23} + \mathrm{c}_{12}\, \mathrm{s}_{23}\, \mathrm{s}_{13}\, \cos(\delta_{13}) \Big) \\
& - m_{u,3}^2 \, \mathrm{s}_{12}\, \mathrm{c}_{23} \Big(\mathrm{s}_{12}\, \mathrm{s}_{23} - \mathrm{c}_{12}\, \mathrm{c}_{23}\, \mathrm{s}_{13}\, \cos(\delta_{13}) \Big) \Bigg] \\
& + \frac{m_{d,2}^2 + m_{d,3}^2}{m_{d,2}^2 - m_{d,3}^2} \Bigg[m_{u,1}^2 \, \mathrm{s}_{12}\, \mathrm{c}_{12}\, \mathrm{s}_{13}\, \cos(\delta_{13}) \\
& + m_{u,2}^2 \, \mathrm{c}_{12}\, \mathrm{s}_{23} \Big(\mathrm{c}_{12}\, \mathrm{c}_{23} - \mathrm{s}_{12}\, \mathrm{s}_{23}\, \mathrm{s}_{13}\, \cos(\delta_{13}) \Big) \\
& - m_{u,3}^2 \, \mathrm{c}_{12}\, \mathrm{c}_{23} \Big(\mathrm{c}_{12}\, \mathrm{s}_{23} + \mathrm{s}_{12}\, \mathrm{c}_{23}\, \mathrm{s}_{13}\, \cos(\delta_{13}) \Big) \Bigg]
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{m_{u,1}^2+m_{u,2}^2}{m_{u,1}^2-m_{u,2}^2}\Bigg[-m_{d,1}^2\,\mathrm{c}_{12}\,\mathrm{c}_{23}\,\mathrm{s}_{13}\Big(\mathrm{c}_{12}\,\mathrm{s}_{23}\,\mathrm{s}_{13}+\mathrm{s}_{12}\,\mathrm{c}_{23}\,\cos(\delta_{13})\Big)\\
&+m_{d,2}^2\,\mathrm{s}_{12}\,\mathrm{s}_{13}\Big(\mathrm{c}_{12}\,\mathrm{c}_{23}^2\,\cos(\delta_{13})-\mathrm{s}_{12}\,\mathrm{s}_{23}\,\mathrm{c}_{23}\,\mathrm{s}_{13}\Big)+m_{d,3}^2\,\mathrm{s}_{23}\,\mathrm{c}_{23}\,\mathrm{s}_{13}^2\Bigg]\\
&+\frac{m_{u,1}^2+m_{u,3}^2}{m_{u,1}^2-m_{u,3}^2}\Bigg[m_{d,1}^2\,\mathrm{c}_{12}\,\mathrm{s}_{23}\,\mathrm{s}_{13}\Big(\mathrm{c}_{12}\,\mathrm{c}_{23}\,\mathrm{s}_{13}-\mathrm{s}_{12}\,\mathrm{s}_{23}\,\cos(\delta_{13})\Big)\\
&+m_{d,2}^2\,\mathrm{s}_{12}\,\mathrm{s}_{23}\,\mathrm{s}_{13}\Big(\mathrm{s}_{12}\,\mathrm{c}_{23}\,\mathrm{s}_{13}+\mathrm{c}_{12}\,\mathrm{s}_{23}\,\cos(\delta_{13})\Big)-m_{d,3}^2\,\mathrm{s}_{23}\,\mathrm{c}_{23}\,\mathrm{s}_{13}^2\Bigg]\\
&+\frac{m_{u,2}^2+m_{u,3}^2}{m_{u,2}^2-m_{u,3}^2}\Bigg[m_{d,1}^2\Big(\mathrm{s}_{23}\,\mathrm{c}_{23}\Big\{\mathrm{s}_{12}^2-\mathrm{c}_{12}^2\,\mathrm{s}_{13}^2\Big\}\\
&+\mathrm{s}_{12}\,\mathrm{c}_{12}\,\mathrm{s}_{13}\Big\{\mathrm{s}_{23}^2-\mathrm{c}_{23}^2\Big\}\cos(\delta_{13})\Big)\\
&+m_{d,2}^2\Big(\mathrm{s}_{23}\,\mathrm{c}_{23}\Big\{\mathrm{c}_{12}^2-\mathrm{s}_{12}^2\,\mathrm{s}_{13}^2\Big\}-\mathrm{s}_{12}\,\mathrm{c}_{12}\,\mathrm{s}_{13}\Big\{\mathrm{s}_{23}^2-\mathrm{c}_{23}^2\Big\}\cos(\delta_{13})\Big)\\
&-m_{d,3}^2\,\mathrm{s}_{23}\,\mathrm{c}_{23}\,\mathrm{c}_{13}^2\Bigg]\Bigg\}
\end{aligned}
\tag{93}
$$

$$
\begin{aligned}
\beta_{13}=\frac{3}{16\pi^2v^2}&\Bigg\{\frac{m_{d,1}^2+m_{d,3}^2}{m_{d,1}^2-m_{d,3}^2}\Bigg[m_{u,1}^2\,\mathrm{c}_{12}^2\,\mathrm{s}_{13}\,\mathrm{c}_{13}\\
&-m_{u,2}^2\,\mathrm{c}_{12}\mathrm{s}_{23}\,\mathrm{c}_{13}\Big(\mathrm{c}_{12}\,\mathrm{s}_{23}\,\mathrm{s}_{13}+\mathrm{s}_{12}\,\mathrm{c}_{23}\,\cos(\delta_{13})\Big)\\
&-m_{u,3}^2\,\mathrm{c}_{12}\,\mathrm{c}_{23}\,\mathrm{c}_{13}\Big(\mathrm{c}_{12}\,\mathrm{c}_{23}\,\mathrm{s}_{13}-\mathrm{s}_{12}\,\mathrm{s}_{23}\,\cos(\delta_{13})\Big)\Bigg]\\
&+\frac{m_{d,2}^2+m_{d,3}^2}{m_{d,2}^2-m_{d,3}^2}\Bigg[m_{u,1}^2\,\mathrm{s}_{12}^2\,\mathrm{s}_{13}\,\mathrm{c}_{13}\\
&-m_{u,2}^2\,\mathrm{s}_{12}\,\mathrm{s}_{23}\,\mathrm{c}_{13}\Big(\mathrm{s}_{12}\,\mathrm{s}_{23}\,\mathrm{s}_{13}-\mathrm{c}_{12}\,\mathrm{c}_{23}\,\cos(\delta_{13})\Big)\\
&-m_{u,3}^2\,\mathrm{s}_{12}\,\mathrm{c}_{23}\,\mathrm{c}_{13}\Big(\mathrm{s}_{12}\,\mathrm{c}_{23}\,\mathrm{s}_{13}+\mathrm{c}_{12}\,\mathrm{s}_{23}\,\cos(\delta_{13})\Big)\Bigg]\\
&+\frac{m_{u,1}^2+m_{u,2}^2}{m_{u,1}^2-m_{u,2}^2}\Bigg[m_{d,1}^2\,\mathrm{c}_{12}\,\mathrm{s}_{23}\,\mathrm{c}_{13}\Big(\mathrm{c}_{12}\,\mathrm{s}_{23}\,\mathrm{s}_{13}+\mathrm{s}_{12}\,\mathrm{c}_{23}\,\cos(\delta_{13})\Big)\\
&+m_{d,2}^2\,\mathrm{s}_{12}\,\mathrm{s}_{23}\,\mathrm{c}_{13}\Big(\mathrm{s}_{12}\,\mathrm{s}_{23}\,\mathrm{s}_{13}-\mathrm{c}_{12}\,\mathrm{c}_{23}\,\cos(\delta_{13})\Big)-m_{d,3}^2\,\mathrm{s}_{23}^2\,\mathrm{s}_{13}\,\mathrm{c}_{13}\Bigg]\\
&+\frac{m_{u,1}^2+m_{u,3}^2}{m_{u,1}^2-m_{u,3}^2}\Bigg[m_{d,1}^2\,\mathrm{c}_{12}\,\mathrm{c}_{23}\,\mathrm{c}_{13}\Big(\mathrm{c}_{12}\,\mathrm{c}_{23}\,\mathrm{s}_{13}-\mathrm{s}_{12}\,\mathrm{s}_{23}\,\cos(\delta_{13})\Big)\\
&+m_{d,2}^2\,\mathrm{s}_{12}\,\mathrm{c}_{23}\,\mathrm{c}_{13}\Big(\mathrm{s}_{12}\,\mathrm{c}_{23}\,\mathrm{s}_{13}+\mathrm{c}_{12}\,\mathrm{s}_{23}\,\cos(\delta_{13})\Big)-m_{d,3}^2\,\mathrm{c}_{23}^2\,\mathrm{s}_{13}\,\mathrm{c}_{13}\Bigg]\Bigg\}
\end{aligned}
\tag{94}
$$

$$
\begin{aligned}
\beta_\delta = \frac{3}{16\pi^2 v^2}\,\sin(\delta_{13}) \Bigg\{ & \frac{m_{d,1}^2 + m_{d,2}^2}{m_{d,1}^2 - m_{d,2}^2} \left[m_{u,2}^2\,\frac{\mathrm{s}_{23}\,\mathrm{c}_{23}\,\mathrm{s}_{13}}{\mathrm{s}_{12}\,\mathrm{c}_{12}} - m_{u,3}^2\,\frac{\mathrm{s}_{23}\,\mathrm{c}_{23}\,\mathrm{s}_{13}}{\mathrm{s}_{12}\,\mathrm{c}_{12}} \right] \\
& + \frac{m_{d,1}^2 + m_{d,3}^2}{m_{d,1}^2 - m_{d,3}^2} \Bigg[m_{u,1}^2\,\frac{\mathrm{s}_{12}\,\mathrm{c}_{12}\,\mathrm{s}_{13}\left\{\mathrm{c}_{23}^2 - \mathrm{s}_{23}^2\right\}}{\mathrm{s}_{23}\,\mathrm{c}_{23}} \\
& + m_{u,2}^2\,\mathrm{s}_{12}\,\mathrm{s}_{23}\frac{\left\{\mathrm{c}_{12}^2 - \mathrm{c}_{23}^2\right\}\mathrm{s}_{13}^2 + \mathrm{c}_{12}^2\,\mathrm{c}_{23}^2\,\mathrm{c}_{13}^2}{\mathrm{c}_{12}\,\mathrm{c}_{23}\,\mathrm{s}_{13}} \\
& + m_{u,3}^2\,\mathrm{s}_{12}\,\mathrm{c}_{23}\frac{\mathrm{s}_{23}^2\,\mathrm{s}_{13}^2 + \mathrm{c}_{12}^2\left\{\mathrm{c}_{23}^2\,\mathrm{c}_{13}^2 - 1\right\}}{\mathrm{c}_{12}\,\mathrm{s}_{23}\,\mathrm{s}_{13}} \Bigg] \\
& + \frac{m_{d,2}^2 + m_{d,3}^2}{m_{d,2}^2 - m_{d,3}^2} \Bigg[m_{u,1}^2\,\frac{\mathrm{s}_{12}\,\mathrm{c}_{12}\,\mathrm{s}_{13}\left\{\mathrm{c}_{23}^2 - \mathrm{s}_{23}^2\right\}}{\mathrm{s}_{23}\,\mathrm{c}_{23}} \\
& + m_{u,2}^2\,\mathrm{c}_{12}\,\mathrm{s}_{23}\frac{\left\{\mathrm{c}_{12}^2 - \mathrm{c}_{13}^2\right\}\mathrm{c}_{23}^2 - \mathrm{s}_{12}^2\,\mathrm{s}_{23}^2\,\mathrm{s}_{13}^2}{\mathrm{s}_{12}\,\mathrm{c}_{23}\,\mathrm{s}_{13}} \\
& + m_{u,3}^2\,\mathrm{c}_{12}\,\mathrm{c}_{23}\frac{\mathrm{c}_{23}^2\,\mathrm{s}_{13}^2 + \mathrm{s}_{23}^2\,\mathrm{c}_{13}^2 + \mathrm{c}_{12}^2\left\{\mathrm{c}_{23}^2\,\mathrm{c}_{13}^2 - 1\right\}}{\mathrm{s}_{12}\,\mathrm{s}_{23}\,\mathrm{s}_{13}} \Bigg] \\
& + \frac{m_{u,1}^2 + m_{u,2}^2}{m_{u,1}^2 - m_{u,2}^2} \Bigg[- m_{d,1}^2\,\mathrm{c}_{12}\,\mathrm{c}_{23}\frac{\mathrm{c}_{12}^2\,\mathrm{s}_{13}^2 + \mathrm{s}_{12}^2\,\mathrm{c}_{13}^2 + \mathrm{c}_{23}^2\left\{\mathrm{c}_{12}^2\,\mathrm{c}_{13}^2 - 1\right\}}{\mathrm{s}_{12}\,\mathrm{s}_{23}\,\mathrm{s}_{13}} \\
& - m_{d,2}^2\,\mathrm{s}_{12}\,\mathrm{c}_{23}\frac{\mathrm{c}_{12}^2\left\{\mathrm{c}_{23}^2 - \mathrm{c}_{13}^2\right\} - \mathrm{s}_{12}^2\,\mathrm{s}_{23}^2\,\mathrm{s}_{13}^2}{\mathrm{c}_{12}\,\mathrm{s}_{23}\,\mathrm{s}_{13}} \\
& + m_{d,3}^2\,\frac{\mathrm{s}_{23}\,\mathrm{c}_{23}\,\mathrm{s}_{13}\left\{\mathrm{c}_{12}^2 - \mathrm{s}_{12}^2\right\}}{\mathrm{s}_{12}\,\mathrm{c}_{12}} \Bigg] \\
& + \frac{m_u 1^2 + m_{u,3}^2}{m_u 1^2 - m_{u,3}^2} \Bigg[- m_{d,1}^2\,\mathrm{c}_{12}\,\mathrm{s}_{23}\frac{\mathrm{s}_{12}^2\,\mathrm{s}_{13}^2 + \mathrm{c}_{23}^2\left\{\mathrm{c}_{12}^2\,\mathrm{c}_{13}^2 - 1\right\}}{\mathrm{s}_{12}\,\mathrm{c}_{23}\,\mathrm{s}_{13}} \\
& - m_{d,2}^2\,\mathrm{s}_{12}\,\mathrm{s}_{23}\frac{-\left\{\mathrm{c}_{12}^2 - \mathrm{c}_{23}^2\right\}\mathrm{s}_{13}^2 + \mathrm{c}_{12}^2\,\mathrm{c}_{23}^2\,\mathrm{c}_{13}^2}{\mathrm{c}_{12}\,\mathrm{c}_{23}\,\mathrm{s}_{13}} \\
& - m_{d,3}^2\,\frac{\mathrm{s}_{23}\,\mathrm{c}_{23}\,\mathrm{s}_{13}\left\{\mathrm{c}_{12}^2 - \mathrm{s}_{12}^2\right\}}{\mathrm{s}_{12}\,\mathrm{c}_{12}} \Bigg] \\
& + \frac{m_{u,2}^2 + m_{u,3}^2}{m_{u,2}^2 - m_{u,3}^2} \left[m_{d,1}^2\,\frac{\mathrm{s}_{12}\,\mathrm{c}_{12}\,\mathrm{s}_{13}}{\mathrm{s}_{23}\,\mathrm{c}_{23}} - m_{d,2}^2\,\frac{\mathrm{s}_{12}\,\mathrm{c}_{12}\,\mathrm{s}_{13}}{\mathrm{s}_{23}\,\mathrm{c}_{23}} \right] \Bigg\}. \qquad (95)
\end{aligned}
$$

$$
\gamma_v = -\frac{2N_c}{16\pi^2 v^2}\left[m_{u,1}^2 + m_{u,2}^2 + m_{u,3}^2 + m_{d,1}^2 + m_{d,2}^2 + m_{d,3}^2 \right] \qquad (96)
$$

$$
\gamma_{u,1} = \frac{3}{16\pi^2 v^2}\left[m_{u,1}^2 - \mathrm{c}_{12}^2\,\mathrm{c}_{13}^2\,m_{d,1}^2 - \mathrm{s}_{12}^2\,\mathrm{c}_{13}^2\,m_{d,2}^2 - \mathrm{s}_{13}^2\,m_{d,3}^2 \right] - 2\frac{\alpha_s}{\pi} \qquad (97)
$$

$$
\begin{aligned}
\gamma_{u,2} = \frac{3}{16\pi^2 v^2} \Bigg[& m_{u,2}^2 - \Big(\mathrm{s}_{12}^2\,\mathrm{c}_{23}^2 + \mathrm{c}_{12}^2\,\mathrm{s}_{23}^2\,\mathrm{s}_{13}^2 + 2\,\mathrm{s}_{12}\,\mathrm{c}_{12}\,\mathrm{s}_{23}\,\mathrm{c}_{23}\,\mathrm{s}_{13}\,\cos(\delta_{13}) \Big)\, m_{d,1}^2 \\
& - \Big(\mathrm{c}_{12}^2\,\mathrm{c}_{23}^2 + \mathrm{s}_{12}^2\,\mathrm{s}_{23}^2\,\mathrm{s}_{13}^2 - 2\,\mathrm{s}_{12}\,\mathrm{c}_{12}\,\mathrm{s}_{23}\,\mathrm{c}_{23}\,\mathrm{s}_{13}\,\cos(\delta_{13}) \Big) m_{d,2}^2 - \mathrm{s}_{23}^2\,\mathrm{c}_{13}^2\,m_{d,3}^2 \Bigg] - 2\frac{\alpha_s}{\pi} \qquad (98)
\end{aligned}
$$

$$\gamma_{u,3} = \frac{3}{16\pi^2 v^2}\Bigg[m_{u,3}^2 - \Big(\mathrm{s}_{12}^2\,\mathrm{s}_{23}^2 + \mathrm{c}_{12}^2\,\mathrm{c}_{23}^2\,\mathrm{s}_{13} - 2\,\mathrm{s}_{12}\,\mathrm{c}_{12}\,\mathrm{s}_{23}\,\mathrm{c}_{23}\,\mathrm{s}_{13}\,\cos(\delta_{13})\Big)\,m_{d,1}^2$$
$$- \Big(\mathrm{c}_{12}^2\,\mathrm{s}_{23}^2 + \mathrm{s}_{12}^2\,\mathrm{c}_{23}^2\,\mathrm{s}_{13}^2 + 2\,\mathrm{s}_{12}\,\mathrm{c}_{12}\,\mathrm{s}_{23}\,\mathrm{c}_{23}\,\mathrm{s}_{13}\,\cos(\delta_{13})\Big)\,m_{d,2}^2$$
$$-\mathrm{c}_{23}^2\,\mathrm{c}_{13}^2\,m_{d,3}^2\Bigg] - 2\frac{\alpha_s}{\pi} \tag{99}$$

$$\gamma_{d,1} = \frac{3}{16\pi^2 v^2}\Bigg[m_{d,1}^2 - \mathrm{c}_{12}^2\,\mathrm{c}_{13}^2\,m_{u,1}^2$$
$$- \Big(\mathrm{s}_{12}^2\,\mathrm{c}_{23}^2 + \mathrm{c}_{12}^2\,\mathrm{s}_{23}^2\,\mathrm{s}_{13}^2 + 2\,\mathrm{s}_{12}\,\mathrm{c}_{12}\,\mathrm{s}_{23}\,\mathrm{c}_{23}\,\mathrm{s}_{13}\,\cos(\delta_{13})\Big)\,m_{u,2}^2$$
$$- \Big(\mathrm{s}_{12}^2\,\mathrm{s}_{23}^2 + \mathrm{c}_{12}^2\,\mathrm{c}_{23}^2\,\mathrm{s}_{13}^2 - 2\,\mathrm{s}_{12}\,\mathrm{c}_{12}\,\mathrm{s}_{23}\,\mathrm{c}_{23}\,\mathrm{s}_{13}\,\cos(\delta_{13})\Big)\,m_{u,3}^2\Bigg] - 2\frac{\alpha_s}{\pi} \tag{100}$$

$$\gamma_{d,2} = \frac{3}{16\pi^2 v^2}\Bigg[m_{d,2}^2 - \mathrm{s}_{12}^2\,\mathrm{c}_{13}^2\,m_{u,1}^2$$
$$- \Big(\mathrm{c}_{12}^2\,\mathrm{c}_{23}^2 + \mathrm{s}_{12}^2\,\mathrm{s}_{23}^2\,\mathrm{s}_{13}^2 - 2\,\mathrm{s}_{12}\,\mathrm{c}_{12}\,\mathrm{s}_{23}\,\mathrm{c}_{23}\,\mathrm{s}_{13}\,\cos(\delta_{13})\Big)\,m_{u,2}^2$$
$$- \Big(\mathrm{c}_{12}^2\,\mathrm{s}_{23}^2 + \mathrm{s}_{12}^2\,\mathrm{c}_{23}^2\,\mathrm{s}_{13}^2 + 2\,\mathrm{s}_{12}\,\mathrm{c}_{12}\,\mathrm{s}_{23}\,\mathrm{c}_{23}\,\mathrm{s}_{13}\,\cos(\delta_{13})\Big)\,m_{u,3}^2\Bigg] - 2\frac{\alpha_s}{\pi} \tag{101}$$

$$\gamma_{d,3} = \frac{3}{16\pi^2 v^2}\Bigg[m_{d,3}^2 - \mathrm{s}_{13}^2\,m_{u,1}^2 - \mathrm{s}_{23}^2\,\mathrm{c}_{13}^2\,m_{u,2}^2 - \mathrm{c}_{23}^2\,\mathrm{c}_{13}^2\,m_{u,3}^2\Bigg] - 2\frac{\alpha_s}{\pi} \tag{102}$$